RESEARCH STATEMENT

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1. Introduction

My research has been primarily focusing on geometric analysis and partial differential equations. The emphasis is on the PDE theory on non-compact manifolds, especially those with singularities, degenerate and singular differential equations, and their applications to applied mathematics.

2. Elliptic and parabolic operators on manifolds with singularities

An interesting problem in geometry is whether one can find a "standard model" in each class of metrics on some manifold, thus reducing topological questions to differential geometric ones. One of the well-known representatives of this kind is the Yamabe problem. On a compact manifold (M, g), this problem aims at finding a metric conformal to g with constant scalar curvature. H. Yamabe's original proof was found erroneous by N.S. Trudinger. N.S. Trudinger [96] and T. Aubin [9] verified Yamabe's conjecture in some restricted cases. Later, the general case was proved by R. Schoen [78].

This problem can be considered also from an evolutionary point of view. Suppose that we already have a metric, as a natural question, one might want to ask how we can "drive" the prescribed metric into a "standard" one, or at least improve it. For example, the Ricci flow tends to "flatten out" or "round out" a manifold depending on its initial "shape". The evolution version of the Yamabe problem is called the Yamabe flow. More precisely, on a compact manifold (M, g_0), the metric $g(t)$ evolves subject to

$$\partial_t g = -R_g g, \quad g(0) = g_0, \tag{2.1}$$

where R_g is the scalar curvature with respect to the evolving metric g. Evolution problems of this kind comprise an important class of so-called geometric flows, or geometric evolution equations.

Compared with our understanding of geometric flows on compact manifolds, the knowledge for the non-compact manifolds case, in particular those with singularities, is indeed limited, mainly because of the failure of many analytic and PDE tools in the non-compact and singular contexts. The development of differential equation theory on these manifolds is thus of significant importance.

To understand part of the difficulty in the analysis on non-compact or singular manifolds, we pick one of the most fundamental operators, the Laplace-Beltrami operator, on conical manifolds as an example. Roughly speaking, a conical manifold M is constructed based on an m-dimensional compact Riemannian manifold $(\mathscr{M}, g)$ with boundary $(\partial\mathscr{M}, g_{\partial\mathscr{M}})$. We can make a collar neighborhood of $\partial\mathscr{M}$ into a conical end by associating it with the degenerate metric $g_c = dt^2 + t^2 g_{\partial\mathscr{M}}$. Now, near the singular end, the Laplace-Beltrami operator Δ_{g_c} can be expressed as

$$\Delta_{g_c} = -t^{-2}((t\partial_t)^2 + (m-2)t\partial_t + \Delta_{g_{\partial\mathscr{M}}})$$

This operator clearly becomes singular towards the conical end. It is well known that analyzing a singular operator is far more difficulty than a uniformly elliptic one, e.g. the Laplace-Beltrami operator on a compact manifold.

2.1. **A historical account of singular analysis.** The investigation of differential operators on manifolds with singularities is motivated by a variety of applications ranging from geometry and topology to applied mathematics, natural sciences and engineering. All of the work is related, in one way or another, to the seminal paper by V.A. Kondrat'ev [55]. There is a tremendous amount of literature analyzing cone differential operators on manifolds with conic ends in terms of suitable parameter-dependent pseudo-differential calculi. The study of these operators was initiated by J. Cheeger [14, 15, 16]; and associated pseudodifferential calculi were devised R.B. by Melrose [65, 66], Plamenevskij [70], and B.-W. Schulze [68, 82, 83, 84]. Many authors have been very active and made remarkable contributions in this field, among them J.B. Gil, T. Krainer and G.A. Mendoza [39, 40, 41, 42, 44], P. Loya [43, 57, 58, 59], J. Seiler, E. Schrohe and S. Coriasco [18, 19, 20, 21, 75, 76, 77, 79, 81], R. Mazzeo and B. Vertman [62, 63]. The amount of research in this field is enormous, and thus it is literally impossible to list all of the achievements.

The study of cone differential operators is a highly technical subject. For example, one may consider a $2l$-th order elliptic differential operator $\mathcal{A}$ with smooth coefficients on a closed manifold M with initial domain $C^\infty(\mathsf{M})$. It is well known that the minimal extension of $\mathcal{A}$ in $L_p(\mathsf{M})$, the closure of $\mathcal{A}$, and the maximal extension of $\mathcal{A}$, that is, all functions in $L_p(\mathsf{M})$ that are mapped into $L_p(\mathsf{M})$ by $\mathcal{A}$, coincide and have the common domain $W_p^{2l}(\mathsf{M})$. This, nevertheless, no longer holds true on manifolds with singularities. This phenomenon has attracted many efforts to study the closed extensions of cone differential operators.

A very active research line in singular analysis is to investigate the property of maximal regularity of cone differential operators. Assume that $X_1 \overset{d}{\hookrightarrow} X_0$ is some densely embedded pair of Banach spaces. Suppose that $dom(\mathcal{A}) = X_1$. Then, $\mathcal{A}$ is said to have L_p-maximal regularity in X_0, if, for any $f \in L_p(\mathbb{R}_+, X_0)$ and $u_0 \in (X_0, X_1)_{1-1/p,p}$, the equation

$$\partial_t u(t) + \mathcal{A}u(t) = f(t), \quad u(0) = u_0$$

has a unique solution $u \in L_p(\mathbb{R}_+, X_1) \cap H_p^1(\mathbb{R}_+, X_0)$, where $(\cdot,\cdot)_{1-1/p,p}$ is the real interpolation method. If we choose to work in a Hölder framework, the counterpart is called continuous maximal regularity. In virtue of theorems by P. Clément and S. Li [17], G. Da Prato and P. Grisvard [22] and S.B. Angenent [7], maximal regularity theory gives rise to local existence and uniqueness for quasilinear or even fully nonlinear equations. This theory has proven itself a powerful tool in the study of nonlinear evolution equations in recent decades. See [1], [6]-[8], [23, 24], [27]-[31], [60, 71, 72, 74] for example. In a series of papers [19, 20, 75], the authors showed that if $\mathcal{A}$ satisfies some special ellipticity conditions, i.e. the invertibility of various symbols in some weighted Sobolev spaces (called Mellin Sobolev spaces), then $\mathcal{A}$ enjoys L_p-maximal regularity for any $1 < p < \infty$. However, if we write a cone differential operator with respect to the conical metric $\mathcal{A}u := \sum_{i=0}^{l} a_i \cdot \nabla_{g_c}^i u$, one can argue that one of these ellipticity conditions, the conormal ellipticity, forces the leading coefficient a_l of the operator to stay away from zero, and thus prohibits the study of degenerate equations. In fact, the practical use of these theorems is confined to operators with non-zero constant coefficients near the singularities. By means of some perturbation techniques, one can relax the regularity assumption on the coefficients and assume that a_i just asymptotically tends to a non-zero constant towards the singularities, cf. [76]; still this apparently limits the application of the theory.

2.2. **A new approach to singular analysis.** Before describing my work in the field of singular analysis, I will first introduce the geometric framework I have been working in.

In 2012, H. Amann [2, 3] introduced a geometric framework, called *singular manifolds*, to study differential equations on non-compact manifolds possibly with singularities. As a starting point, he first defined a class of possibly non-compact manifolds, called *uniformly regular Riemannian manifolds*. Roughly speaking, a manifold (M, g) is *uniformly regular* if its differential structure is induced by an atlas whose local patches are of comparable sizes, and all transition maps and curvatures have uniformly bounded derivatives. In the boundaryless case, this definition coincides with manifolds of bounded geometry. See [26]. (M, g) is a *singular manifold* if $(\mathsf{M}, g/\rho^2)$ is *uniformly regular* for some $\rho \in C^\infty(\mathsf{M}, (0,\infty))$, i.e., it is conformally *uniformly regular*. Conventionally, we write M, g, ρ as a 3-tuple $(\mathsf{M}, g; \rho)$. This concept covers the class of manifolds with cylinder, cone, edge and corner ends. To the best of my knowledge, the last three are all the types of singularities that have been studied in other literature on singular analysis.

The motivation to introduce the framework of *singular manifolds* is multifold.

Firstly, many efforts have been made to generalize research in singular analysis to more complicated types of singularities, e.g., edge and corner ends. However, for higher order singularities, the corresponding algebra becomes significantly more complicated. Therefore, a natural question to ask is whether one can find a general approach, which is less sensitive to the geometric structure near the singularities, to singular analysis. The framework of *singular manifolds*, as we can observe from its definition, does not depend on the order of the singularities or the specific geometric structure near the singularities.

Secondly, utilizing the uniform structure of *singular manifolds*, function space theory, including those pertaining to interpolation, embedding and trace theorems, can be well established therein, cf. [2, 3]. Although most function spaces, like Sobolev-Slobodeckii spaces, can be defined invariantly on non-compact manifolds, these crucial theorems for PDE analysis do not hold in this generality. For instance, neither interpolation nor trace theorem holds for the Mellin Sobolev spaces. Therefore, the framework of *singular manifolds* is particularly useful in the study of nonlinear PDE on manifolds with singularities.

My work in singular analysis has mainly focused on establishing the maximal regularity theorem for differential operators, including degenerate and singular ones, on *singular manifolds*. In two of my recent papers [89, 91], I have proved the following L_p-maximal regularity result.

Theorem 2.1 ([89, 91]). *Suppose that $(\mathsf{M}, g; \rho)$ is a C^2-singular manifold satisfying $\rho \leq 1$, and*

$$|\nabla\rho|_g \sim \mathbf{1}, \quad \|\Delta\rho\|_\infty < \infty \quad \textit{near the singularities.} \tag{2.2}$$

Let $\lambda' \in \mathbb{R}$, and $\lambda \in [0,1) \cup (1,\infty)$. Furthermore, assume that the differential operator

$$\mathscr{A}u := -\mathrm{div}(a_2 \mathrm{grad} u) + a_1 \cdot \nabla u + a_0 u$$

satisfies

$$a_2 = \rho^{2-\lambda}, \quad a_1 \in W^{1,\lambda}_\infty(\mathsf{M}, T_{\mathsf{M}}), \quad a_0 \in L^\lambda_\infty(\mathsf{M}).$$

Then $\mathscr{A}$ has L_p-maximal regularity in $L^{\lambda'}_p(\mathsf{M})$ with dom$(\mathscr{A}) = W^{2,\lambda'-\lambda}_p(\mathsf{M})$ for any $1 < p < \infty$.

Here the symbol $\sim$ always denotes Lipschitz equivalence; and the space $W^{s,\vartheta}_p(\mathsf{M})$ can be understood as some weighted W^s_p-space with weight ρ^ϑ. To keep this article at a reasonable length, I would like to refer the reader to [2] for the precise definitions of these spaces. It is worthwhile mentioning that one of the key steps in the proof for the above theorem was inspired by an idea in a paper[54] of J.J. Kohn and L. Nirenberg.

To understand the value of this theorem, first, one can verify that the geometric assumption (2.2) is satisfied by cone and edge manifolds (but not restricted to these classes). Secondly, the diffusion coefficient $a_2 = \rho^{2-\lambda}$ is asymptotically zero if $0 < \lambda < 2$, and tends to ∞ as $\lambda > 2$. Hence the operators in Theorem 2.1 include not only those discussed in the cone differential operator theory, that is, the operators with coefficients asymptotically tending to a non-zero constant, but it can also tackle both degenerate and singular operators. Besides, the extra regularity assumption $a_2 = \rho^{2-\lambda} \in C^2(\mathsf{M})$ can actually be reduced to assuming a_2 belong to some proper C^1-space by means of a standard perturbation argument, as shown by the application in [49]. Indeed, in a manuscript in preparation [93], I am able to replace the condition $a_2 = \rho^{2-\lambda}$ by a more elegant one:

- When $\lambda \neq 2$, $a_2 \in bc^{1,\lambda-2}(\mathsf{M})$, $|\nabla a_2^{\frac{1}{2-\lambda}}| \sim \mathbf{1}$, $a_2 > C\rho^{2-\lambda}$ for some $C > 0$;
- When $\lambda = 2$, $a_2 = C + w$ with $w \in BC^{1,\vartheta}(\mathsf{M})$ with $\|w\|_\infty < C$ and $\vartheta < 0$, for some $C > 0$.

Here $bc^{k,\vartheta}(\mathsf{M})$ and $BC^{1,\vartheta}(\mathsf{M})$ are some weighted little Hölder and C^1-spaces, cf. [90, 92, Section 2.2].

In another project [90], I established the continuous maximal regularity for degenerate or singular differential operators of arbitrary even order acting on sections of vector bundles.

Theorem 2.2 ([90]). *Suppose that $(\mathsf{M}, g; \rho)$ is a C^{2l}-singular manifold, and a $2l$-th order linear differential operator*

$$\mathscr{A}u := \sum_{i=0}^{2l} a_i \cdot \nabla^i u$$

satisfies some proper ellipticity condition, and a_i are $bc^{s,0}$-continuous, then $\mathscr{A}$ has continuous maximal regularity in $bc^{s,\vartheta}(\mathsf{M}, T^\sigma_\tau \mathsf{M})$ for any $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and $\vartheta \in \mathbb{R}$.

Here $T^\sigma_\tau \mathsf{M} := T\mathsf{M}^{\otimes\sigma} \otimes T^*\mathsf{M}^{\otimes\tau}$. To explain the ellipticity condition in Theorem 2.2, for simplicity's sake, we take $\sigma = \tau = 0$, i.e. $T^\sigma_\tau \mathsf{M} = \mathbb{R}$. Then, in this simplified case, the ellipticity condition can be stated as

$$a_{2l} \cdot (-i\xi)^{\otimes 2l} \sim \rho^{2l} |\xi|^{2l}_{g^*}, \quad \xi \in T^*\mathsf{M},$$

where g^* is the cotangent metric induced by g. Thus, when $l = 1$, Theorem 2.2 complements Theorem 2.1 for the case $\lambda = 2$. I would like to mention that similar results to Theorem 2.2 in an L_p-framework have been obtained by H. Amann [4].

These results in [4, 89, 90, 91] laid the theoretic foundation for further investigation into many evolution problems on manifolds with singular ends.

2.3. **The Yamabe flow.** On a closed manifold (M, g_0), the short time existence of the Yamabe flow (2.1) is a consequence of the compactness of (M, g_0) and a fixed point theorem argument. The convergence of the flow to a constant scalar curvature metric was proved by R.G. Ye [100] for the scalar negative, scalar flat and locally conformally flat scalar positive cases. The more complicated cases were completed by H. Schwetlick and M. Struwe [85], S. Brendle [11, 12] for dimension $3 \leq m \leq 5$, and with technical assumptions on the Weyl curvature in dimensions $m \geq 6$.

Nevertheless, the theory for the Yamabe flow on non-compact manifolds is far from being settled. Even local well-posedness is only established for restricted situations. To the best my knowledge, the only known results for non-compact complete manifolds were established by L. Ma and Y. An in [61] for initial metrics with bounded Ricci and scalar curvature and by the author's work with G. Simonett [87] for initial metrics with possibly unbounded curvatures.

The research on the more challenging case of the Yamabe flow on manifolds with singularities mainly focuses on dimension two, in which case the Yamabe flow coincides with the well-known Ricci flow up to a constant multiple. The Ricci flow on a manifold (M, g_0) possibly with singularities reads as

$$\partial_t g = -2\mathrm{Ric}_g, \quad g(0) = g_0. \tag{2.3}$$

Here Ric_g is the Ricci curvature tensor with respect to g. In [52, 64, 99], the authors analyzed the solution to the Ricci flow on manifolds with isolated conical singularities, which starts from a conic metric and preserves the original singular type for all time.

A remarkable feature of the Ricci flow, and thus the Yamabe flow, with singular initial metrics, e.g. conical metrics, in dimension 2 is the loss of uniqueness of the solution. Except for the aforementioned family of singularity preserving solutions, another important class of solutions is the instantaneously complete solutions. A Ricci flow $(\mathsf{M}, g(t))$ is called instantaneously complete if for all $t > 0$ $(\mathsf{M}, g(t))$ is complete. Note that a *singular metric* g of a *singular manifold* (M, g), and thus a conical metric, is incomplete, which indicates the loss of uniqueness of solutions to (2.3). This class of solutions originally appeared in a paper by E. DiBenedetto and D.J. Diller [25], and was later simplified and generalized by G. Giesen and P. Topping. The research on this topic has drawn increasing interest during recent years. See [36, 37, 38, 94, 95, 101] for related results. This class of solutions will be discussed in more detail in Section 5.

The non-uniqueness issue addressed above, thus, suggests that a careful choice of proper functional setting is paramount in the analysis of the Yamabe flow and related problems on incomplete manifolds. The first result of the singular Yamabe flow in higher dimension is obtained by E. Bahuaud and B. Vertman [10]. They studied the Yamabe flow on a compact manifold with asymptotically simple edge singularities. Their proof for short time existence is based on a careful analysis of the mapping property of the heat operator between Hölder spaces defined with respect to an edge metric.

Let $g = u^{\frac{4}{m-2}} g_0$ for some positive conformal factor u, where m is the dimension of the manifold. Then (2.1) is equivalent to

$$\partial_t u = u^{-\frac{4}{m-2}} (\Delta_{g_0} u - \frac{m-2}{4(m-1)} R_{g_0} u), \quad u(0) = \mathbf{1}. \tag{2.4}$$

Here $\mathbf{1}$ always denotes the constant 1 function. Based on Theorem 2.1, in the author's paper [91], short time existence and uniqueness result for the Yamabe flow similar to those in [10, 52, 64, 99] was established.

Theorem 2.3 ([91]). *Suppose that* $(\mathsf{M}, g_0; \rho)$ *is a* C^2*-singular manifold satisfying* (2.2), *and* $p > m+2$. *We assume that there exists some* $\varepsilon \in (\frac{2}{p}, \infty)$ *such that* $\rho^{\varepsilon - \frac{m+2}{p}} \in L_p(\mathsf{M})$ *and* $R_{g_0} \in BC^{\infty, 2-\varepsilon}(\mathsf{M})$. *Then* (2.4) *has a unique positive solution for some* $J = [0, T]$ *with* $T > 0$

$$u \in L_p(J; W_p^{2, -\frac{m+2}{p}}(\mathsf{M}, \mathbb{R})) \cap H_p^1(J; L_p^{2-\frac{m+2}{p}}(\mathsf{M}; \mathbb{R})) \oplus \mathbb{R}_{\mathsf{M}}.$$

Here $\mathbb{R}_{\mathsf{M}}$ is the set of all functions that are constants on each component of M. Suppose that $(\mathscr{M}, g)$ is a compact manifold with boundary. Let $(\mathsf{M}, g_0) := (\mathscr{M} \setminus \partial\mathscr{M}, g|_{\mathsf{M}})$. Then Theorem 2.3 holds true for (M, g_0) and gives a unique local in time W_p^2 solution to (2.1). Moreover, this solution retains the type of singularity throughout the existence interval. The reader may also refer to [90, Section 4.4] for a related result.

It is worthwhile pointing out that these solutions, or equivalently the specific functional settings, in [10, 52, 64, 91, 99] actually indicate certain asymptotic behaviors at the singular ends; and thus we can make sense of some kind of boundary condition. This "completes" the missing information caused by the incompleteness of the initial data.

3. Degenerate and singular differential equations

For many decades, the study of degenerate and singular differential equations has played an indispensable role in modelling numerous real-world phenomena, like population growth, heat transportation, fluid dynamics. For instance, in the diffusion of a substance in some medium, or heat flow in a material, or diffusion of a population in a habitat, the nonhomogeneity of the medium is expressed by the spatial dependence of the diffusion coefficient with its possible vanishing at some points. For example, a certain composite material can block the heat flow at a certain point, or the migration of small mammal species can degenerate due to environmental heterogeneity and barriers. The equations governing these phenomena are usually expressed in terms of degenerate parabolic equations.

Various approaches have been utilized to tackle degenerate and singular differential equations, e.g. Hardy-Poincaré type inequalities, Harnack type inequalities, variational method, and entropy solutions. However, in comparison with non-degenerate or non-singular equations, very little is known for degenerate or singular ones.

The theory of *singular manifolds* provides a new perspective of the degenerate and singular equations. To understand the intrinsic relationship between singular analysis and degenerate/singular equations, we look at the baby example of the heat equation in the open m-disk, $\mathbb{B}^m(0,1)$,

$$\partial_t u - \Delta u = f, \quad u(0) = u_0. \tag{3.1}$$

This seems to be an ill-posed problem because of the lack of boundary information. However, if we consider (3.1) from the *singular manifolds* point of view, and think of the boundary $\mathbb{S}^{m-1}$ as a singular point. This is a well-stated problem. Indeed, after a conformal change of metric, the Poincaré disk given by

$$(\mathbb{B}^m(0,1), g_h) := (\mathbb{B}^m(0,1), g_m/(1-d^2)^2),$$

with g_m is the m-dimensional Euclidean metric and $d(x) := |x|$, is a manifold with bounded geometry. With respect to the metric g_h, the principal symbol of $-\Delta$ is

$$-(i\xi, i\xi)_{g_m} = |\xi|^2_{g_h^*}/(1-d^2)^2$$

for any cotangent field ξ. This is a singular elliptic operator. Hence, (3.1) can be considered as Cauchy problem of a singular parabolic equation. Therefore, the singularity (incompleteness) of the metric can be rescaled to become the degeneracy/singularity in an equation, and conversely the degeneracy or singularity in an equation can be removed by shifting the degeneracy or singularity to the metric.

Based on my work mentioned in Section 2.2, in a series of papers [89]-[91], the maximal regularity theory for degenerate and singular differential equations of various rate of degeneracy and singularity has been established. For instance, we consider the second order differential operator

$$\mathcal{A}u := -\mathrm{div}(a_2 \mathrm{grad} u) + a_1 \cdot \nabla u + a_0.$$

This operator may exhibit both boundary or interior degeneration and singularity. Denote the distance of x to the set of degeneracy/singularity by $\mathsf{d}(x)$. The diffusion coefficient is assumed to satisfy $a_2 \sim \mathsf{d}^\alpha$ near the set of degeneracy/singularity. When $\alpha < 2$, L_p-maximal regularity theory has been proved for $\mathcal{A}$, see [89, 91]. While $\alpha \geq 2$, the continuous maximal regularity for $\mathcal{A}$ was obtained in [90]. In the latter case, the results for higher order differential operators in non-divergence form $\mathcal{A}u := \sum_{i=0}^{2l} a_i \cdot \nabla^i u$ also hold.

To illustrate the value of these theorems, we look at a parabolic equation with interior degeneration

$$y_t - \partial_x(u(x)\partial_x y_x) = f, \quad y(0) = y_0, \qquad (x,t) \in (0,L) \times (0,T), \tag{3.2}$$

with suitable boundary conditions. u can degenerate in the interior of $(0,L)$, e.g. $u = (x - x_0)^\alpha$ for $x_0 \in (0,L)$. This problem has been studied extensively for u degenerating at both the boundary and the interior in the past decades. In the interior case, to the best my knowledge, results are only obtained in an L_2-semigroup framework, cf. [33]-[35], which prohibits the study of similar problems of nonlinear type in the higher dimensional case. The methods utilized in the cited references also have the drawback of losing a precise characterization of the domain of the operator, which creates another barrier to applying these result to nonlinear equations. This disadvantage is also shared by the study of many boundary degeneration problem, cf. [32, 97].

Nevertheless, in conjunction with a fixed point theorem argument, as mentioned above, the maximal regularity results in [89]-[91] can be easily applied to solve nonlinear differential equations.

4. **Image processing**

As an application of the maximal regularity theory for degenerate or singular equations, in a project wtih P. Guidotti, we studied a spatial regularization of a celebrated PDE based de-noising model by P. Perona and J. Malik [69], the Perona-Malik equation, which can be formulated as follows. Let $\mathsf{Q}^N = (-1,1)^N$ with $N = 1,2$.

$$\begin{cases} \partial_t u - \operatorname{div}\left(\dfrac{1}{1+|\nabla u|^2}\nabla u\right) = 0 & \text{in} \quad \mathsf{Q}^N \times (0,\infty); \\ \partial_\nu u = 0 & \text{on} \quad \partial\mathsf{Q}^N \times (0,\infty); \\ u(0) = u_0 & \text{in} \quad \mathsf{Q}^N. \end{cases} \tag{4.1}$$

Numerical experiments show that the desired effect is obtained; more precisely, the solution $u(t)$ is a sharper image than the input u_0 for small positive t, and sharp edges are preserved well in a long run. This is in stark contrast with the theoretical ill-posedness of (4.1), which has been called the Perona-Malik paradox [53]. An easy computation shows that when $|\nabla u(x,t)|$ is large across a hypersurface, then (4.1) is not parabolic near (x,t), but it engenders a backward heat flow, which is well known to be highly unstable. However, the occurrence of sharp gradients, for example, when u_0 is close to a piecewise constant function, is exactly the most interesting case in applications.

A large amount of efforts have been contributed to ameliorate (4.1) by means of regularized or relaxed models. In a series of papers [45]-[47], P. Guidotti and J. Lambers have proposed and analyzed several novel spatial regularizations of (4.1). Among them, one model is implemented by using the fractional derivative.

$$\partial_t u - \operatorname{div}\left(\frac{1}{1+|\nabla^{1-\varepsilon} u|^2}\nabla u\right) = 0, \quad u(0) = u_0 \tag{4.2}$$

with $\varepsilon \in (0,1)$. Periodic or Dirichlet/Neumann boundary conditions can be imposed. Well-posedness of this model has been proved for Hölder continuous initial data in [45]. It is of great interest in both applications and mathematical theory to study characteristic functions and linear combinations thereof, which readily leads to degenerate parabolic equations. Numerical simulations shows that the characteristics of the Perona-Malik equation, including the edge preserving properties, are well preserved in (4.2). See [45, 46]. Characteristic functions of smooth sets are the steady states of (4.2). However, in a recent paper [48], non-trivial dynamical behavior for piecewise constant initial data has been observed to occur in numerical experiments as the parameter ε exceeds the threshold value 1/2. Solutions to (4.2) typically tend to a trivial state for $\varepsilon > 1/2$. Similar numerical results can also be found in [46]. This non-uniqueness phenomenon reveals that the mathematical analysis of degenerate equations like (4.2) heavily relies on the choice of functional setting.

In a recent paper [49] with P. Guidotti, we proved the well-posedness and stability of equilibria of (4.2) for a class of discontinuous initial data, which is large enough to include linear combinations of characteristic functions of smooth sets. More precisely, under periodic boundary conditions and suitable assumptions on p, ϑ, the following theorem holds.

Theorem 4.1 ([49]). *Suppose that H is a linear combination of characteristic functions of disjoint C^3-domains Ω_i in $\mathring{\mathsf{Q}}^N$. Let $\Gamma = \cup_i \partial\Omega_i$. Given any $u_0 = H + w_0$ with*

$$w_0 \in W^{2-2/p,\vartheta+\frac{2\varepsilon}{p}}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma),$$

equation (4.2) has a unique solution

$$u \in L_p(J, W^{2,\vartheta}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma)) \cap H^1_p(J, L^{\vartheta+2\varepsilon}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma)) \oplus \mathbb{R}_{\mathsf{Q}^N\setminus\Gamma}.$$

for some $J := [0,T]$ with $T = T(u_0) > 0$. Moreover, if $\varepsilon > 1 - 2/p$, then H is a stationary solution to (4.1) and attracts all solutions which are initially $W^{2-\frac{2}{p},\frac{2\varepsilon(1-p)}{p}}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma)$ close to H exponentially fast.

$\mathbb{R}_{\mathsf{Q}^N\setminus\Gamma}$ denotes the set of all periodic functions that are constants on each component of $\mathsf{Q}^N \setminus \Gamma$; and $W^{s,\vartheta}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma)$ stands for the space of periodic $W^{s,\vartheta}_p(\mathsf{Q}^N \setminus \Gamma)$ functions.

5. Instantaneously regularized solution to parabolic equations with interior degeneracy

The solutions in Theorem 4.1 keep the edges (which can be considered as the singular type of these solutions) of the inputs whenever they exist. A challenging question is how to capture the occurrence of the blurring effects in the solutions to (4.2), or the linearized equation, observed in the numerical simulations in [46, 48].

To understand the subtlety of this problem, we recall the existence of instantaneously complete solutions to the Ricci flow with singular initial metric. P. Topping stated the following example in [94].

Example 5.1. *Suppose that (M, g_0) is the standard 2 dimensional open disk. Then there exists a Ricci flow $g(t)$ such that $(\mathsf{M}, g(t))$ is complete for all $t > 0$ and $g(0) = g_0$.*

As commented in Section 2.3, the preservation of singular types of the solutions to geometric flows with singular metrics can be viewed as a complement for the missing "boundary condition". With the absence of this asymptotic information, an incomplete Ricci flow can immediately push points within finite distance to "infinity". This surprising phenomenon, to some extend, can be abstractly understood by looking at the fast diffusion equation governing the conformal factor of the evolving metric. The instantaneous completion of the metric will immediately blow up the conformal factor at "the boundary". The fast diffusion equation then evolves into a strongly degenerate equation with some boundary blow-up effect, which eliminates the need for "boundary information". The reader may refer to [13, 67, 91] for related results. From this point of view, it can be understood that with insufficient information from the input, the equation immediately attempts to complete itself.

In Theorem 4.1, roughly speaking, the preservation of the edges provides some alternative "boundary information" at the edges. After removing this alternative information, the above observation suggests that the equation (4.2) should tend to complete itself. A natural way to do so is: if the evolving image in (4.2) is immediately smoothed out, then (4.2) evolves into a non-degenerate parabolic equation and thus the behavior of the solution will be determined therefrom. This is exactly the blurring effect indicated on the numerical level, and thereby proposes the existence of instantaneous regularized solutions to degenerate equations like (4.2).

In a ongoing project [50] with P. Guidotti, we have proved the existence of such solutions to the linearization of (4.2). More precisely, we linearize the nonlinearity of (4.2) at a function H, a linear combination of characteristic functions of disjoint C^3-domains, as in Theorem 4.1. Then the resulting equation

$$\partial_t u - \operatorname{div}(a_\varepsilon \nabla u) = 0, \quad u(0) = H + w_0, \tag{5.1}$$

where $a_\varepsilon \sim \mathsf{d}^{2-2\varepsilon}$ with $\varepsilon > 1/2$ near Γ and w_0 is the same as in Theorem 4.1, has a solution

$$u \in L_p(\mathbb{R}^+, W^{2,\vartheta}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma)) \cap H^1_p(\mathbb{R}^+, L^{\vartheta+2\varepsilon}_{p,\pi}(\mathsf{Q}^N \setminus \Gamma)) \oplus \mathbb{R}_{\mathsf{Q}^N\setminus\Gamma}.$$

On the other hand, we can also show that there is a global solution $\bar{u}$ to (5.1) such that

$$\bar{u}(t) \in C^1(\mathsf{Q}^N) \text{ for all } t > 0.$$

Recently, after acquiring some refined estimates, we believe that the existence of instantaneously regularized solutions should hold for the original nonlinear equation (4.2) as well.

Conjecture 5.2. *Under the same conditions as in Theorem 4.1, for $\varepsilon > 1/2$, (4.2) has a solution*

$$u \in C^\infty((0,T) \times \mathsf{Q}^N), \quad T > 0.$$

These findings convince us that the exploration of instantaneously regularized solutions should be a promising research line for many a de-noising model, e.g. (4.2), as well as some other degenerate parabolic problems, since our construction of instantaneously regularized solution does not depend on the specific structure of the equation (4.2). Another open question in this direction is whether the instantaneously regularized solution is unique in some appropriate sense.

6. **Some other research projects**

6.1. **Some past projects.** (1) During my Ph.D. training, with my Ph.D. advisor G. Simonett, we have established in [87] continuous maximal regularity theory for elliptic operator of arbitrary even order acting on sections of vector bundle on *uniformly regular Riemannian manifolds*. Recently, a complete maximal regularity theory in some anisotropic Sobolev-Slobodeckii framework and little Hölder framework was established in [5]. The Fourier analytic method used to prove this result is novel, which can be used to treat all function spaces via one approach and is unified for both elliptic and parabolic equations.

(2) In [88], I developed a technique relying on a family of parameter-dependent diffeomorphism, maximal regularity theory, and the implicit function theorem to prove regularity of solutions to geometric flows. This technique was later generalized in [73] to show regularity of free boundary problems. As applications of this technique, the temporal analyticity of the Ricci flow and the regularity of the free boundary to a thermodynamically consistent two-phase Stefan problem with surface tension have been established. See [86, 88] for more related results. The analyticity of the Ricci flow was also proved by B. Kotschwar [56] via a different approach.

(3) Recently, in [92], I have proved the global existence of solutions to the porous medium equation on a classed of *singular manifolds*, including but not restricted to conic manifolds. The short time existence of solution to the porous medium equation on conic manifolds was established by N. Roidos and E. Schrohe in [76].

(4) In a project with T. Huang and C. Wang [51], we have proved that the harmonic heatflow in dimension 2 with weak boundary anchoring admits a unique global weak solution; and results on the boundary and interior bubbling are also obtained.

6.2. **Ongoing projects and future research plan.** The discovery of instantaneously regularized solutions to (5.1) has launched my research in a new direction. The construction of such solutions to nonlinear degenerate equations is not merely valuable for the study of the de-nosing model (4.2); using similar construction to prove the existence of instantaneously complete solutions to other problems, like the Yamabe flow in dimension larger than 2, is also an interesting and promising direction.

The maximal regularity theorems stated in Section 2 should be valuable assets to the study of many problems on deforming manifolds driven by their curvatures as well as free boundary problems with initial singularities. Through some informal analysis, I believe that we can prove the existence of local solutions to the mean curvature flow and the surface diffusion flow when the initial manifolds has isolated singularities. This result may also pave the way for studying free boundary problems, e.g. the Stefan problem, with initial singular interface.

One of my long term research plan is to study the PDE theory on manifolds with evolving singularities. This direction, while technically very challenging, is not simply a branch of singular analysis, meanwhile it has also close connection with free boundary problems and image processing. For example, the moving boundary in a free boundary problem can be considered as the evolving singularities, and such theory is also capable of capturing and analyzing the moving edge phenomena appeared in some image processing models.

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